REPORT No. 251

APPROXIMATIONS FOR COLUMN EFFECT IN AIRPLANE WING SPARS

By EDWARD P. WARNER and MAC SHORT

Massachusetts Institute of Technology

REPORT No. 251

APPROXIMATIONS FOR COLUMN EFFECT IN AIRPLANE WING SPARS

By Edward P. Warner and Mac Short

SUMMARY

The significance attaching to "column effect" in airplane wing spars has been increasingly realized with the passage of time, but exact computations of the corrections to bending moment curves resulting from the existence of end loads are frequently omitted because of the additional labor involved in an analysis by rigorously correct methods. The present report, prepared for publication by the National Advisory Committee for Aeronautics, represents an attempt to provide for approximate column effect corrections that can be graphically or otherwise expressed so as to be applied with a minimum of labor. Curves are plotted giving approxiate values of the correction factors for single and two bay trusses of varying proportions and with various relationships between axial and lateral loads. It is further shown from an analysis of those curves that rough but useful approximations can be obtained from Perry's formula for corrected bending moment, with the assumed distance between points of inflection arbitrarily modified in accordance with rules given in the report.

The discussion of general rules of variation of bending stress with axial load is accompanied by a study of the best distribution of the points of support along a spar for various conditions of loading.

GENERAL NATURE OF PROBLEM

The vital significance of column effect in a slender beam carrying both lateral load and compression has long been realized, and formulæ for dealing with the case of the laterally loaded strut and calculating the equivalent bending moment at various points of such a member have been available for many decades. In civil and mechanical engineering structures, however, the occurrence of beams stressed in flexure and carrying also a compressive load approximating to the "Euler load," or that which would cause failure by lateral instability with lateral load entirely lacking, is rare. Only with the coming of the airplane did the use of members working under such conditions become a commonplace of design, and only then did the need for accurate and complete means of calculating them become acute.

The first efforts to treat "column effect," or the modification of the bending moment diagram by the amount of compression moments dependent on deflection were made through approximations, of which a great number have been devised. One or more are given in every work on the strength of materials, and Newell lists (Reference 1) eight approximate formulæ, all intended to serve the same purpose and giving widely varying results in some cases. Not until 1916 did a complete and rigorously accurate method of calculation of continuous beams under combined load become available. (Reference 2.)

With the development of the "Berry Method" or "Exact Method" or "Generalized Theorem of Three Moments," as it is variously called, the problem of column effect passes into a new stage. Accurate calculations could be made, subject to the usual assumptions about homogeneity of material and the absence of discontinuities of section, and the question then became one of whether or not it was worth while to go through with the somewhat elaborate

processes involved in any particular case. In many instances it is not worth while, or is not considered by the designer to be so, and although the generalized equation has had a steadily increasing use in practice its employment is still far from universal.

Taking all this into account, the desirability of having some reliable means of approximating the column effect in advance is obvious. Such a trustworthy means, if one can be secured, will show in some cases that column effect is unimportant and that no check calculations including allowance for it are necessary, while in those instances where it is of importance the inclusion of a preliminary allowance for its magnitude will decrease the liability of having to make two or more tentative spar designs with a calculation for each one.

The difficulty of making exact calculations is not only inherent in the form of the equations, which require for their solution perhaps triple the time needed for the analysis of a continuous beam by ordinary methods, especially in view of the fact that some of the results from the generalized form come out as the relatively small difference of two very large quantities and the number of significant figures that have to be preserved through the preliminary operations is therefore greater than would be necessary for a corresponding degree of accuracy if the equation were left in its familiar form. The labor of a solution is further increased by the fact that the generalized method is only a check, which can not be applied until a preliminary design of the spar section has been made. It is necessary to work the problem through once in order to arrive at that preliminary design and then to work it through again to be sure that the preliminary selection would fit the requirements. Furthermore, in case the check shows the section first chosen to be inadequate or overstrong still another complete set of calculations must be made after the redesign.

The stresses in an airplane wing spar depend on certain properties of the spar section, the wing truss form, and the loading, and for any given general form of truss the number of such properties exerting an influence is definite and small. In a two-bay truss with the spars continuous over three points of support and hinged at their inner ends, for example, the bending and direct stresses depend on the length of each of the three bays of the spar, on the moment of inertia, section modulus, and area of its cross-section, and on the lateral load per unit of length and on the compressive or tensile load applied. The compressive load, however, is itself dependent only on the lateral load, the distribution of supports along the spar, and the gap (if it be a biplane that is under consideration), and the variation of stress with change of linear dimensions in a truss in which geometric similarity is preserved, follows simple and well understood laws. The number of variables can therefore be materially reduced if all factors involving linear dimensions are converted to a constant overall length of spar as a basis of comparison, and coefficients of bending moment figured without regard to column effect can be plotted, for example, in terms of the percentage of total length of spar in the inner bay and the percentage in the overhang as the only two independent variables. Since the inner bay, outer bay, and overhang sum to 100 per cent, any other pair from among those three might be selected instead of the inner bay and overhang as the basic quantities governing the magnitude of the bending moment coefficients.

A complete solution without column effect has been made and published for two-bay wing trusses of proportions extending over the full range of probable design practice. (Reference 3.) It is only a little more difficult to do the same thing with column effect taken into account, and the work has been carried through for two typical wing truss forms, the single bay with overhang and the two-bay truss treated in Technical Report No. 214 (Reference 3), the bending moment at the inner end of the spar being assumed zero in all cases.

SINGLE-BAY TRUSS

When a spar is continuous over only two supports and is pinned at the inner end, the bending moments at the support are obviously independent of any column action, that at the inner end being zero while that at the strut is governed only by the length and loading of the overhanging portion of the spar. The distribution of moment in the bay, however, is directly affected by

the compression column moments of magnitude Py, where y is the deflection of the spar after all loads are on it, having to be added to or subtracted from the moments due to lateral load, the sign of the correction of course depending on the signs of the lateral load moment and of the deflection. In general, it is expected that the correction to be applied to the maximum value of the moment in the middle of a bay will be positive, as a spar running over only two supports commonly deflects in the direction of the lateral load, and that gives an additive column correction. It will, in fact, always do so unless the ratio of length of overhang to length of bay is exceptionally great.

The formula for the maximum value of a bending moment attained between supports is given by Berry as:

$$M_{max.} = \frac{w}{\mu^2} - \left(\frac{w}{\mu^2} - \frac{M_A + M_B}{2}\right) \sec \mu x \sec \alpha$$

where x is the distance from the point of maximum moment to the middle point of the bay, and is in turn defined by:

$$\tan \mu x = \frac{M_A - M_B}{M_A + M_B - \dfrac{2w}{\mu^2}} \cot \alpha$$

and the other symbols are:

$P =$ compressive load
$E =$ modulus of elasticity of material
$I =$ moment of inertia of cross section
$w =$ lateral load per unit length of spar

$$\mu = \sqrt{\frac{P}{EI}}$$

$l =$ length of bay of spar

$$a = \frac{l}{2}$$

M_A and $M_B =$ moments at ends of bay

$$\alpha = a\mu = \frac{l}{2}\sqrt{\frac{P}{EI}}.$$

In the particular case under consideration M_B is 0, and the form becomes somewhat simplified, being most conveniently expressible for most calculations in the shape:

$$\frac{w}{\mu^2}(1 - \sec \mu x \sec \alpha) + \frac{M_A}{2}\sec \mu x \sec \alpha$$

it will be observed in this formula that each term varies as wl^2 if α is kept constant. It is possible to consider ordinary bending moment stress charts, such as were given in Report No. 214 (Reference 3), therefore as representing the special case in which α equals $0°$ (no compressive load), and similar sets of charts could be developed for any other value of α. For a given value of that quantity, in other words, coefficients of bending moment can always be written independently of the absolute values of the lateral loading, the dimensions of the truss, and the dimensions of the spar section, varying only with the distribution of the struts along the spar, or in this particular case with the placing of the single strut.

The calculation has been made for five cases, with the length of the effective overhang 22.2, 30, 35, 37.5, and 40 per cent of the total effective overall length, the true length having been reduced for the tip loss allowance. The bending moment at the strut, all results being reduced to a total spar length of 100 inches and to a loading of 1 pound per inch of length, would then be 246, 450, 612, 703, and 800 pounds inches in the five cases, while the maximum bending moment in the bay with no column effect would on similar assumptions be 638, 408 266, 200, and 139 pounds inches.

Making similar calculations of moment in the bay for the five cases with various values of α ranging from 15° to 85°, the values found to change are tabulated below and plotted in Figure 1.

The statement of α as an angle is of course a mathematical fiction. It is actually a pure number, which becomes equal to $\pi/2$ when the point of collapse as a column is reached, but
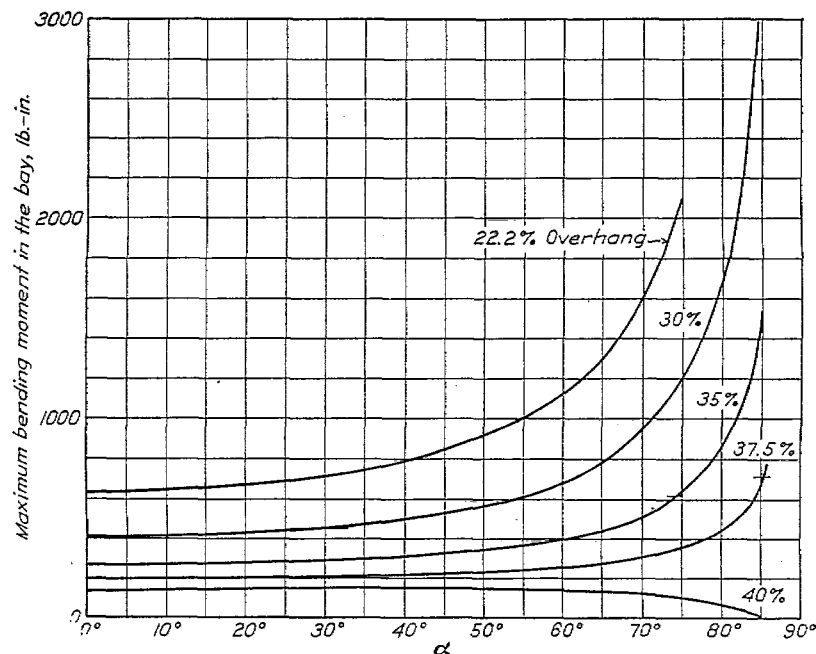

Fig. 1

since the development of the theory brings in certain terms which can be best expressed by trigonometric functions, it is sometimes convenient to state the variable as an angle. The direct solution, $\alpha = \frac{l}{2}\sqrt{\frac{P}{EI}}$, of course gives the angle in radians.

	Moments in bay					$\dfrac{\text{Moment in bay}}{\text{Moment without column effect}}$				
	Per cent length of overhang									
	22.2	30	35	37.5	40	22.2	30	35	37.5	40
0°	638	408	266	200	139	1.00	1.00	1.00	1.00	1.00
15°	653	416	270	--------	140	1.02	1.02	1.02	--------	1.01
30	710	446	288	211	141	1.11	1.09	1.08	1.06	1.01
45	843	530	323	--------	142	1.32	1.30	1.22	--------	1.02
60	1,124	680	397	261	139	1.76	1.67	1.49	1.31	1.00
75	2,090	1,196	623	--------	112	3.28	2.93	2.34	--------	.81
80	--------	1,706	850	431	78	--------	4.18	3.20	2.16	.56
85	--------	3,284	1,536	671	6	--------	7.98	5.77	3.36	.04

As the computations in making up this table were never carried beyond four significant figures, the last digit in the moment figures is somewhat uncertain but the probable error is not in excess of 1%.

In Figure 1 the moments are plotted directly, while Figure 2 shows the ratios of correction factor required by column effect. It will be observed that there is comparatively little difference among the first four curves in Figure 2, suggesting the possibility of using a standard curve of the correction factor to be applied to moments in the middle of the bay, and the value of α for all single bay wing trusses with overhang lengths of less than 37 per cent

provided α does not exceed about 45°. Figure 3 is plotted with the same ordinates as Figure 2, but with the abscissæ changed to the ratio $\frac{P}{P_e}$, where P_e is the Euler load, $\frac{\pi^2 EI}{l^2}$. This ratio is equal to $\left(\frac{\alpha}{90}\right)^2$.

In Figure 1, on the curves corresponding to the 30, 35, and 37.5 per cent, overhangs the value of the moment at the strut has been indicated by a short cross line. Even with an overhang as long as 37.5 per cent, approaching a semicantilever form, the moment in the bay supersedes that at the strut as the critical element in the calculation when the end load is more than 90 per cent of that under which the spar would collapse as a simple column pin-jointed at its ends.

While it would be possible, as just noted, to plot a single curve of moment correction factors for single-bay trusses which would hold within a maximum error of 10 per cent or 15 per cent for all cases likely to be met with in practice, except those with overhangs so long that they would ordinarily be described and thought of as semicantilever types, it would be better either to continue the use of several such curves as those in Figure 3 and to interpolate for the par-
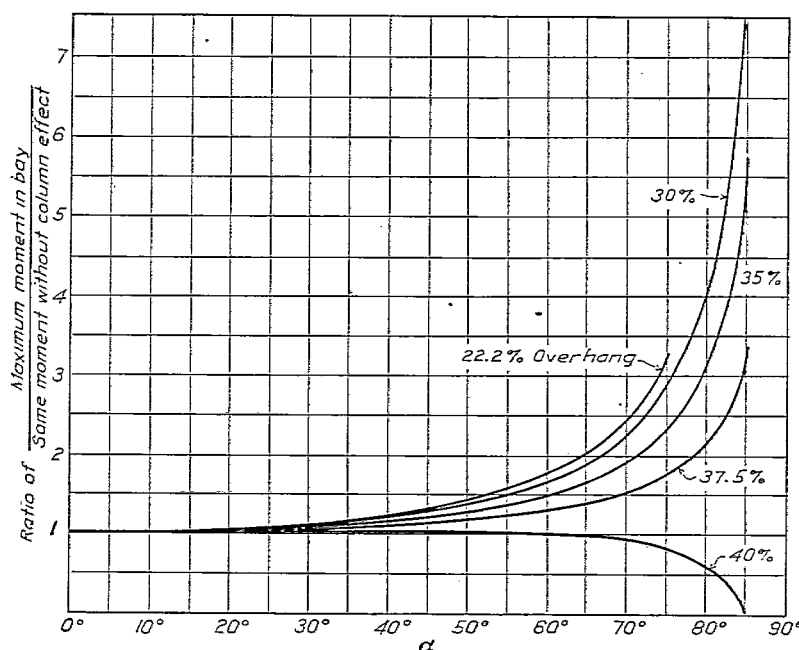

FIG. 2

ticular length of overhang under consideration, or, alternatively, to find some general approximate method, either emperical or rational, which comes sufficiently close to fitting all cases.

An approximation much used in such cases is the Perry formula. (Reference 4.)

$$M' = \frac{M \times P_e}{P_e - P}$$

where P is the end load actually carried, P_e the Euler load, M the moment under lateral load alone, and M' the moment with allowance for column effect. Since the formula was devised for application to struts carrying a lateral load and pin-jointed at their ends, it is manifestly unfair to allow nothing for stiffening of the spar by continuity past one or both of the ends of the bay, and in actual use the factor P_e is commonly replaced by P_e', equal to $\frac{\pi^2 EI}{l'^2}$, l' is the distance between the points of inflection rather than that between the ends of the bays. That distance itself, of course, varies with the length of overhang, a curve of $\frac{l'}{l}$

for the single-bay spar with hinged inner end being plotted against percentage length of overhang in Figure 4.

Unfortunately, the points of inflection do not remain fixed, but move outward toward the ends of the bay as the column effect becomes increasingly important. The writer has made it
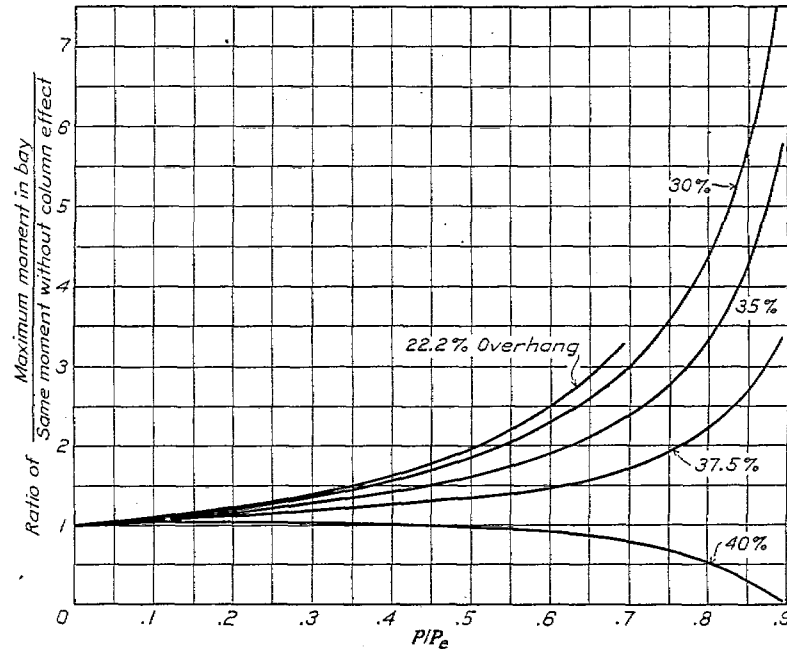

FIG. 3

a practice to allow for this shift empirically by basing the Euler load in the Perry formula neither on l nor l', but on a fictitious length $l' + n(l - l')$. The coefficient n is, of course, fractional and varies with the type of truss under consideration. For the single-bay truss with the spar pin-jointed at the inner end experience shows that a value of 0.8 for n gives best results.
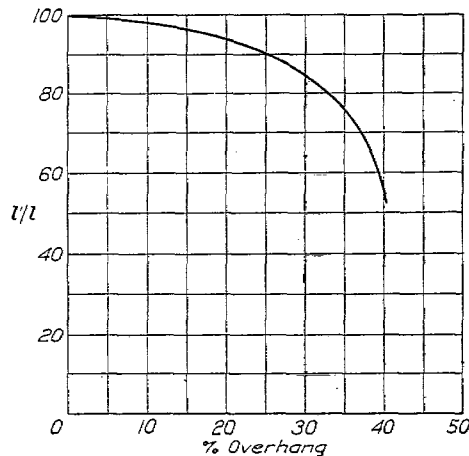

FIG. 4

The Perry formula correction factors have been calculated on that basis, and the ratio of the correct maximum bending moment in the bay to that approximately determined by the formula is plotted in Figure 5. The 40 per cent overhang has been omitted in both cases, as it is manifest that no correction factor is needed in that case, and any factor or method which increases the value of the moment will be worse than none at all.

It is evident that the Perry formula gives excellent results in general. The values for the maximum moment derived by its use are correct within 5 per cent for all values of α up to 68° with a 22 per cent overhang, 76° with 30 per cent, 44° with 35 per cent, and 32° with 37.5 per cent. Most of the cases met with in practice will show values of α lower than these, and it can therefore be said in general that the Perry formula gives approximations sufficiently accurate for ordinary use without any exact calculation except when the overhang is unusually
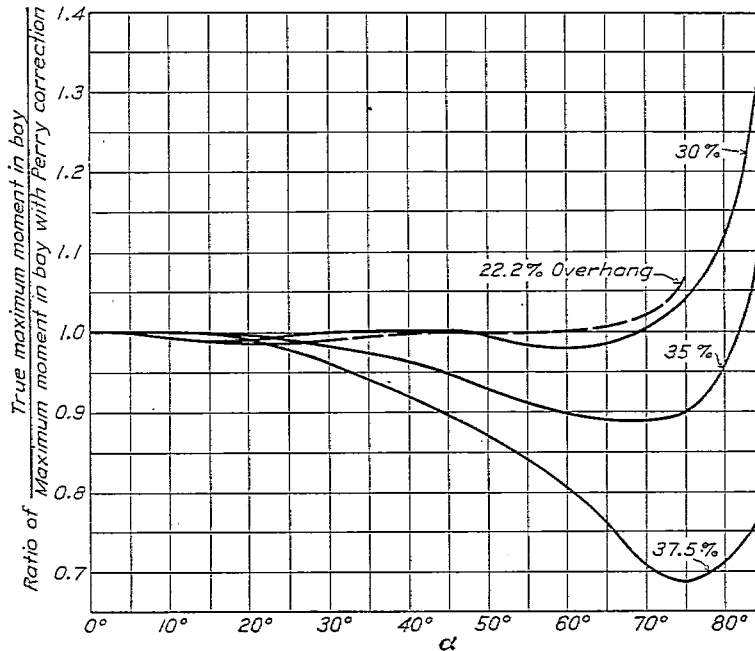

FIG. 5

long and the compression in the bay large. Exact figures can be read off, of course, by interpolation between the curves in Figure 3 if that is desired.

The sudden change of form of the curve of variation of maximum bending moments with compression in the bay in going from a 37.5 to 40 per cent overhang is startling at first sight. It is, however, entirely logical, and it would be expected that there would be a very sharp, if not absolutely discontinuous, passage from beams in which the moment increases very rapidly
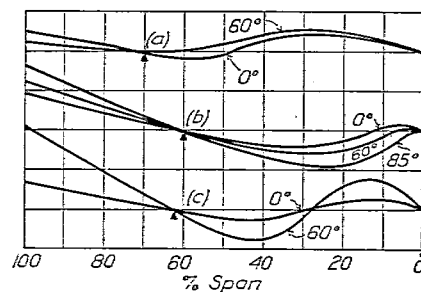

FIG. 6

at large values of α to those in which it decreases with corresponding rapidity in the same region. The reason is that when α approaches very closely to 90° the compression moments become the controlling factor, and whether those moments act with or against the effect of lateral load in the middle of the bay depends on the form of the initial deflection curve, which, in turn, is governed by the length of overhang. Figure 6 illustrates diagrammatically the general form of deflection curves that would be found with 30 and 40 per cent overhang, respectively, and the variation of the curve with changing α in the two cases. Manifestly, when α is

very large it is almost certain that either the part of the beam which was initially deflected above the datum line or that which was deflected below will have taken control, and the column moments will have thrown all the deflections in one direction.   Such a condition as is indicated in the third diagram in Figure 6 is conceivable, but most unstable and unlikely.

In order to illustrate the amount of work involved in an accurate calculation for one overhang the complete figures for the case of 40 per cent overhang are tabulated below:

<div align="center">ILLUSTRATIVE CALCULATION</div>

<div align="center">Length of bay=60″
Length of overhang=40″
$w=1$ lb./in. $M_A=800$ lb. in.</div>

	15	30	45	60	75	80	85
α (deg.)	15	30	45	60	75	80	85
α (rad.)	0.26180	0.52360	0.78540	1.04720	1.30900	1.39627	1.48353
sec α	1.03528	1.15470	1.41421	2.00000	3.86371	5.75877	11.47380
csc α	3.86371	2.00000	1.41421	1.15470	1.03528	1.01543	1.00382
$\mu = \left(\dfrac{\alpha}{30}\right)$	.008727	.017453	.026180	.034907	.046542	.046542	.049451
$\dfrac{1}{\mu^2}$	13130.2	3282.92	1459.02	820.68	525.25	461.65	408.93
M_A csc α	3090.97	1600.00	1131.37	923.76	828.22	812.34	803.06
M_A sec α	828.22	923.76	1131.37	1600.00	3090.97	4607.02	9179.00
$\left(\dfrac{1}{\mu^2}-\dfrac{M_A}{\ell}\right)$ sec $\alpha = (A)$	13179.2	3328.90	1497.68	841.36	483.95	355.01	102.46
$\dfrac{M_A\ \csc \alpha}{\ell\ \sec\left(\dfrac{1}{\mu^2}-\dfrac{M_A}{\ell}\right)}=(\tan \mu x)$	.11724	.2403	.3777	.5489	.8556	1.1442	3.919
μx (deg.)	6° 41′	13° 31′	20° 41′	28° 46′	40° 33′	48° 51′	75° 41′
μx (rad.)	.1167	.2359	.3610	.5020	.7078	.8526	1.3209
x	13.37	13.52	13.79	14.38	16.22	18.32	26.71
sec μx	1.0069	1.0285	1.0689	1.1407	1.3160	1.5190	4.0450
sec μx sec α	1.0424	1.1875	1.5117	2.2814			
$M_{max.}=\left((A)\ (\sec \mu x) -\dfrac{1}{\mu^2}\right)$	139.7	140.9	141.8	139.0	111.6	77.8	5.5

LOCATION OF STRUT

The most economical strut location in a single-bay truss with untapered wing spars, considering only spar weight, is obviously that which makes the moment at the strut equal to the maximum in the bay.   The percentage length of overhang to be used to conform with that specification is plotted against α in Figure 7, and it will be observed that the variation in strut location with change of α is very slow.   To determine the exact value for any particular case α should be found in terms of other and more general properties of the airplane.
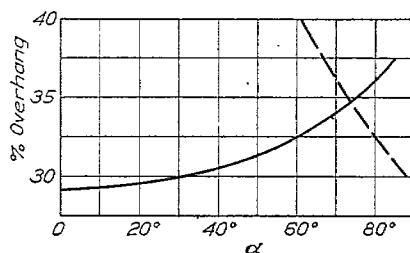

<div align="center">Fig. 7</div>

In the particular, and very common, case of the biplane with upper and lower wings of equal size, and with the loading on the lower wing assumed to be 90 per cent of that on the upper, the compression in the upper spar is independent of strut location, and is always:

$$P=\frac{1.9\ Wl^2}{2G}=\frac{Wl^2}{1.05G}$$

where P is the compression, w the load on the spar in pounds per unit of length, l the total effective length of the spar (with the correction for tip loss taken off), and G the gap.   The moment of inertia of the spar by the familiar formula for bending stress is equal to:

$$I=\frac{Mc}{f_b}$$

c being the distance from the neutral axis to the outermost fiber. On the assumption that the bending moment in the bay is to be equal to that at the strut and that the latter can therefore be taken as one of the worst conditions, I becomes for a symmetrical spar:

$$\frac{wdl_o{}^2}{4f_b}$$

l_o being the length of the overhang and d the depth of the spar. Then, l_b being the length of the bay,

$$\frac{P}{EI}=\left(\frac{l}{l_o}\right)^2\times\frac{3.8f_b}{GdE}$$

$$\alpha=\frac{l_b}{2}\sqrt{\frac{P}{EI}}=\sqrt{\frac{l^2{}_b\,l^2}{l_o{}^2}\times\frac{f_b}{1.05GdE}}=\sqrt{\frac{l_b}{l_o}\frac{f_b}{E}\frac{l^2}{1.05Gd}}$$

f_b can be determined in terms of the ultimate total stress.

$$\frac{f_b}{f_t}=\frac{\dfrac{Mc}{I}}{\dfrac{Mc}{I}+\dfrac{P}{A}}$$

On the assumption, previously discussed elsewhere (Reference 5) that the radius of gyration of an airplane wing spar section is 0.36 of the depth,

$$\frac{c}{I}=\frac{1}{.26dA}$$

$$M=\frac{wl_o{}^2}{2}\qquad\qquad P=\frac{wl^2}{1.05G}$$

$$\frac{f_b}{f_t}=\frac{1}{1+.48\dfrac{d}{G}\left(\dfrac{l}{l_o}\right)^2}$$

The product of $\frac{d}{G}$ and $\left(\frac{l}{l_o}\right)^2$ practically always lies between 0.64 and 1.33, corresponding to a range of $\frac{f_b}{f_t}$ of from 0.61 to 0.78. Either in spruce or in duralumin, $\frac{f_b}{E}$ will then be between $\frac{1}{285}$ and $\frac{1}{340}$, with the smaller values going with a short overhang.

Taking the factors entering into the expression for α separately, $\frac{l_b}{l_o}$ may vary 1.5 to 3.0, $\frac{l}{G}$ from 2.2 to 4.6, and $\frac{l}{d}$ from about 25 to 53. For any particular values of those quantities α can be calculated from the formula, and the value so found should be equal to that which makes the moments at the strut and in the bay equal for the overhang length under consideration.

The method of determining optimum overhang can best be demonstrated by a particular problem. $\frac{l}{G}$ will be taken as 3.5, $\frac{l}{d}$ as 40, and a series of overhang lengths tried.

$\frac{l_o}{l}$	$\frac{l_b}{l_o}$	$\frac{f_b}{f_t}$	$\frac{f_b}{E}$	α (rad.)	α (deg.)
.30	2.33	.682	$\frac{1}{305}$	1.54	88°
.35	1.86	.745	$\frac{1}{279}$	1.28	73°
.40	1.50	.793	$\frac{1}{262}$	1.07	61°

These points have been plotted in the dotted line in Figure 7, and the intersection of the two curves defines the best effective overhang at just under 35 per cent (equivalent, with the usual deduction of one-fifth or one-sixth of a chord length from the end to allow for tip losses, to about 40 per cent true overhang).

The solution of other problems of a similar nature gives the data for Figure 8, where curves are plotted to show the best length of overhang in terms of $\frac{l}{d}$ and $\frac{l}{G}$. The extreme range of useful values, it will be noted, is from 31 to 39 per cent.
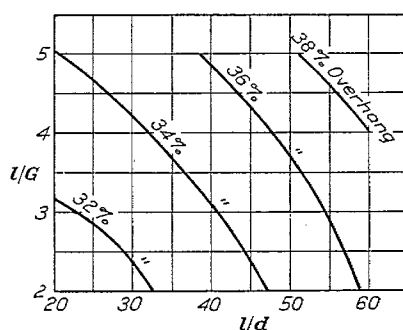

FIG. 8

TWO-BAY TRUSSES

The same general method can be used for investigating the magnitude of column effect in wing spars continuous over three supports, although of course the number of variables steadily increases with increasing number of bays. Such a study has been made for a particular case of the two-bay spar pin-jointed at its inner end.

The trusses selected for calculation in this case have the proportions listed below:

No.	Length inner bay	Outer bay	Overhang
	Per cent	Per cent	Per cent
I	32.5	45	22.5
II	29.5	54.6	15.9
III	33.3	38.5	28.2
IV	37.1	42.9	20
V	27.1	50	22.9
VI	40	45	15
VII	35	45	20
VIII	30	45	25
IX	40	50	10
X	31.7	48.3	20

The calculations for the first case, and for the first alone, were carried through completely for seven different values of α ranging from 15° to 85°. The general form of the curve of variation of moments with changing compression in the spar having thus been determined, only a few values were selected for each of the other cases. The inner and outer bays of course have different values of α. The larger of the two values was in every case used as a key number for indexing. In other words, when α is mentioned in a tabulation or plot it is, unless otherwise specified, taken from the bay having the largest α. That is true even when the specific quantity tabulated or plotted in terms of α relates directly to the other bay, the one with the smaller α. In most two-bay spars α is larger in the inner bay than in the outer, and it is therefore in the inner bay that the column effect is most critical, but if for any reason the ratio of length of outer bay to inner is made especially great that condition is changed, and it is between the inner and outer struts that the spar comes nearest to failure as a simple column. Among the cases here discussed, α was greater in the outer bay than in the inner only in Nos. II, V, and VIII.

A beam continuous over three supports and pinned at the inner end of course has a zero bending moment at that point, and the moment at the other end support, corresponding to the outer strut, is dependent only upon the properties of the overhanging portion and not on the

compression in the bays.   The moment at the middle support, however, varies with the compression—a fact which is not taken into account by any of the ordinary approximations for column action.   It is manifest that in general the bending moment at the intermediate support of such a beam will be increased by compression in the bays, as the general direction of the deflection is the same as that of the lateral load, and the application of compressive force tends therefore to spring the mid-point of the beam up away from its restraint at the support with increased force.   The reaction at the middle support is therefore increased and the bending moment must increase in absolute value likewise.

The way in which the magnitude of any increase depends on α and on the distribution of the supports of the spar is best shown by tabulation and plotting of particular examples.   The values of M_B (B being the middle support) for the nine cases previously described are tabulated below, while curves of M_B against α are plotted for five of them in Figure 9.
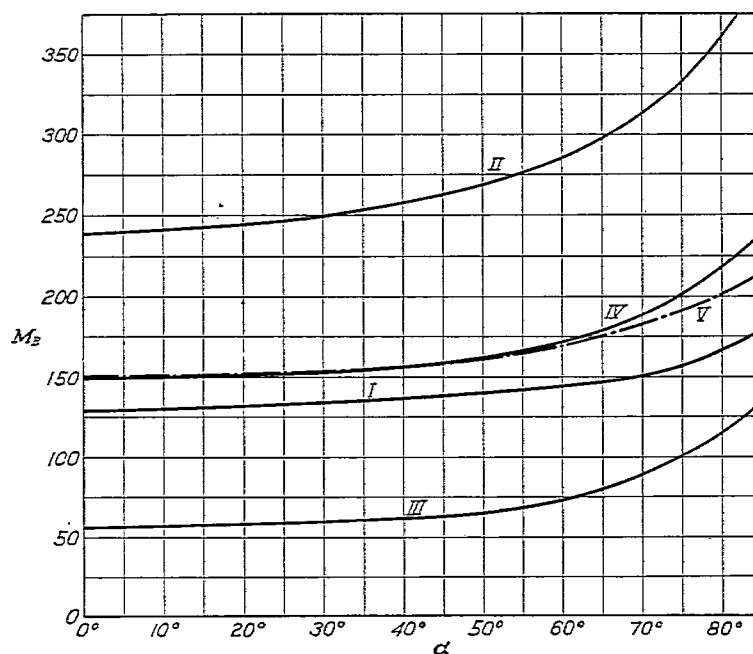

FIG. 9

No.	α							
	0°	15°	30°	45°	60°	75°	80°	85°
I	129	130	132	137	145	158	164	179
II	239	------	248	------	285	------	359	------
III	57	------	59	------	73	------	115	------
IV	149	------	154	------	172	------	218	------
V	150	------	154	------	170	------	201	------
VI	198	------	------	------	224	------	------	------
VII	153	------	------	------	173	------	------	------
VIII	97	------	------	------	111	------	------	------
IX	248	------	------	------	273	------	------	------

Figure 9 having shown the variation of the absolute values of the moment, Figure 10 represents relative variation for ratio of the moments of middle support for various values of α to the corresponding figure with α equal to zero (column effect absent).   The curves are plotted against α and each one represents one of the typical cases calculated.

To correlate these results and make it possible to predict probable values of M_B as far as the proportions are different from these here tried, a contour chart was constructed with curves representing equal values of the relative column effect or percentage correction to M_B at a given α.   The form of the chart showed that the correction factors were substantially a factor

of the percentage length of the outer bay, varying through a range of only 3 or 4 per cent with varying distribution of the remaining spar length between the inner bay and the overhang. Curves of correction factor to be applied to M_B for various values of α have there-
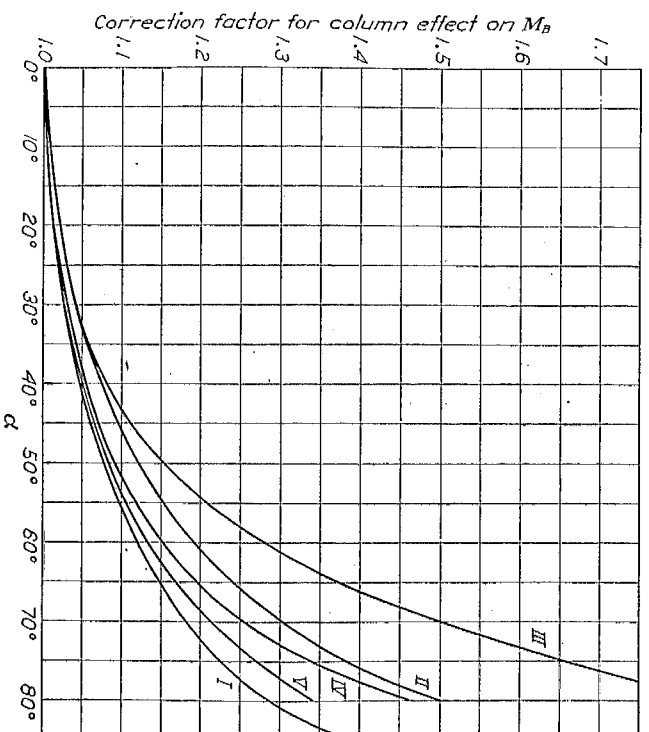

Fig. 10

fore been plotted against the length of outer bay in Figure 11. They are necessarily somewhat rough in form, being based on such a small number of points, but they should make it possible to estimate M_B with an error not in excess of 5 per cent for any probable strut location as long
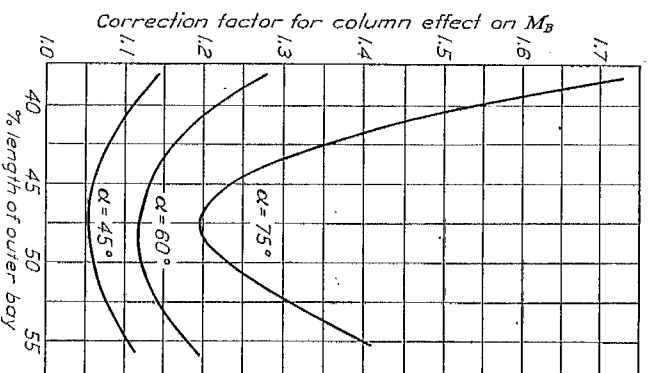

Fig. 11

as α does not exceed 65°. It is not likely to pass that point by much, if at all, with the gap-chord ratios and aspect ratios used in practice.

A preliminary estimate of M_B by the use of these curves serves two purposes. In the first place it will often make it possible to design a spar directly with no explicit calculation by

the exact method for the particular dimensions in question. Secondly, it greatly reduces the time that sometimes has to be given to repeated trials to determine the correct section and correct values of α for the outer and inner bays. Manifestly any change of M_B also changes the compression in the outer bay and the value of α in that bay. That makes astonishingly little difference so long as α in the outer bay is smaller than that in the inner bay. An arbitrary reduction of α in the outer bay in Case I by one-third changed the bending at the middle support by less than 2 per cent, so long as α did not exceed 82° in the inner bay. In the general case in which the inner bay is the critical one, recalculation to allow for change in compression in the outer bay is therefore unnecessary, but when the outer bay is the larger value of α those changes become very serious and anything that can be done to facilitate their prediction in advance and without any repeated trials, as these charts facilitate it, tends toward increased accuracy and economy of time.

MOMENTS WITHIN THE BAYS

The maximum values of the moments within the inner and outer bays, as well as those of M_B, the moment at the inner strut, have been calculated directly for a number of illustrative cases, including all except Nos. VI, VIII, and IX of the 10 already tabulated and any additional one, No. XI, introduced to cover the case in which the values of α in the inner and outer bay are very slightly unequal.

As a first step in the systematic treatment of the results of these calculations 60° has been selected as a standard value of α and the figures for the seven strut distributions tried are tabulated herewith for that value. As before, the bay in which α is the largest, whether inner or outer, is always taken as the basis of rating, and the value of α is considered as fixed at 60° in that bay.

Case	Lengths			Bay having largest value of α (α in that bay = 60°)	Corresponding α in other bay
	Inner bay	Outer bay	Overhang		
I	32.5	45.0	22.5	Inner	54.7°
II				Outer	48.5
III				Inner	45.8
IV				do	36.2
V				Outer	46.0
VII	35.0	45.0	20.0	Inner	48.4
X	31.7	48.3	20.0	Identical	60.0
XI	32.5	47.5	20.0	Inner	57.1

Case	M_{max} in bay where α is largest	M_{max} in that bay with no column effect	Ratio: $\dfrac{M_{max.}\,(\alpha=60°)}{M_{max.}\,(\alpha=0°)}$	$M_{max.}$ in other bay	$M_{max.}$ with no column effect	Ratio: $\dfrac{M_{max.}}{M_{max.}\,(\alpha=0°)}$
I	106	75	1.41	66	66	1.00
II	273	191	1.43	7	23	.30
III	183	119	1.54	−36	−4	
IV	154	104	1.48	45	58	.78
V	140	113	1.25	26	33	.79
VII	119	85	1.40	79	77	1.03
X {Inner	62	56	1.10			
{Outer	165	108	1.53			
XI	80	63	1.21	129	91	1.42

The fact most strikingly evidenced by the table is that column effect is always of negligible importance or actually negative in the bay, having the smaller α, unless the two bays are very close together in that respect. The correction factor for the moment in that bay exceeds unity only when the inner strut is within 4 per cent of the length of the spar from the position which would make the two values of α exactly equal.

The explanation is of course essentially the same that served to account for the falling off of mid-bay moments with increasing compression in a single-bay spar with a long overhang. The deflection being largest in that bay in which α is largest, that portion of the spar "takes control" as the compression is increased and forces a steady decrease of the deflection in the other bay. An actual negativing of the column effect and decrease of mid-bay moments in the bay with the smaller α result in many cases. The nature of the effect is graphically shown in Figure 12, where the deflection curves for the spar covered by Case I are plotted for three values of α (in the inner bay). These curves, it should be noted, are not diagrammatic sketches like those of Figure 6, but are the result of actual deflection calculations and accurately represent the change of the form of the elastic curve for a particular spar as the compression varies, the lateral load remaining constant.

The comparatively slight change of the deflection curve with changing α when that quantity is below 60° and the much more rapid change as 90° is approached is strikingly shown, as is the manner in which the inner bay "takes control" when the compression is very large so that the spar is finally forced down below the line of supports everywhere in the outer bay, giving a negative column effect on the moments there, even though the line of the spar continues to show a double reversal of curvature within the bay. Even under the largest end loads the deflections in the bays are small compared with that at the tip of the overhang, which to be sure is unusually long in this particular spar.
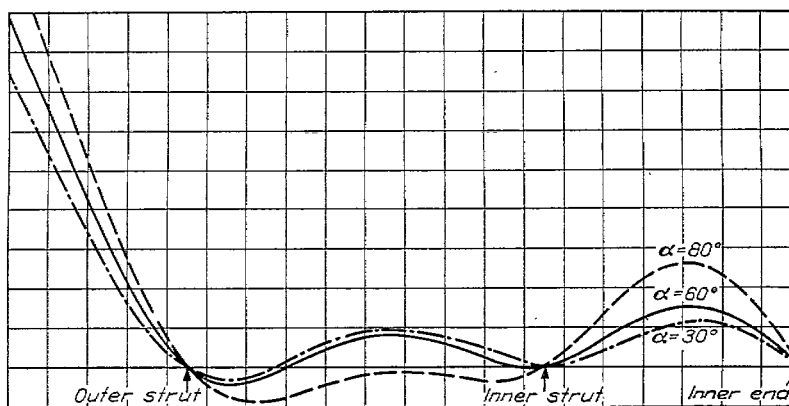

FIG. 12

In the bay where α is largest, and where column effects are therefore uniformly positive, the correction factors on mid-bay moments range from 1.10 to 1.54 in the group of illustrative problems considered. There are hardly enough points to plot a satisfactory contour chart of correction factors in terms of the dimensions of the truss, and therefore the figures can most easily be systematized by relating them to the factors given for corresponding conditions by Perry's approximate formula, exactly as was done in examining the moments in the single-bay spar.

Again the value of n, the coefficient expressive of the proportion of the residual length of the bay by which the points of inflection are to be assumed shifted apart in figuring P_c' for the Perry correction factor, has been determined empirically, and in this case the best figure to use was found to be 0.2 in place of the 0.8 determined for the single bay. It is natural that n should be reduced, since the moments at the ends of the single bay are independent of the compression, while in a two-bay spar the increase of M_B with increasing compression tends to shift the points of inflection closer together and so to neutralize in part the direct effect of the column effect within the bay in relocating those points.

The table below includes the actual correction factors, reproduced from the previous table, the factors calculated by the Perry formula, and the ratio of the two.

Correction factors are to be applied to the maximum moments within the bay in which column effect is greatest when α in that bay is 60°.

Case	Bay considered	True correction factor	Correction factor by Perry formula ($n = .2$)	$\dfrac{\text{True factor}}{\text{Perry factor}}$
I	Inner	1.41	1.41	1.00
II	Outer	1.43	1.36	1.05
III	Inner	1.54	1.61	.96
IV	Inner	1.48	1.55	.95
V	Outer	1.25	1.26	.99
VII	Inner	1.40	1.40	1.00
X	{Inner	1.10	1.32	.83
	{Outer	1.53	1.27	1.20
XI	Inner	1.21	1.34	.90

It will be observed that the true and approximate correction factors never differ by more than 6 per cent unless the values of α in the inner and outer bays differ from each other by less than 8 per cent. If α is the same in the two bays, however, the approximation goes badly astray. To show where this critical region of equal α lies the curve in Figure 13 has been drawn, showing
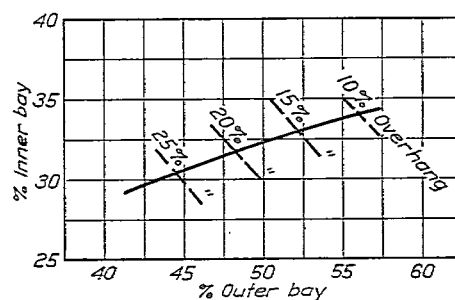

FIG. 13—Proportions of truss for equal α in two bays

the relationship that must exist among the three parts of the length of the spar in order that α may be exactly the same in the two bays.

All of the conclusions so far drawn, and all of the discussion except that on the deflection curves of Figure 12, have related directly to a single particular value of α. In Case I, for which
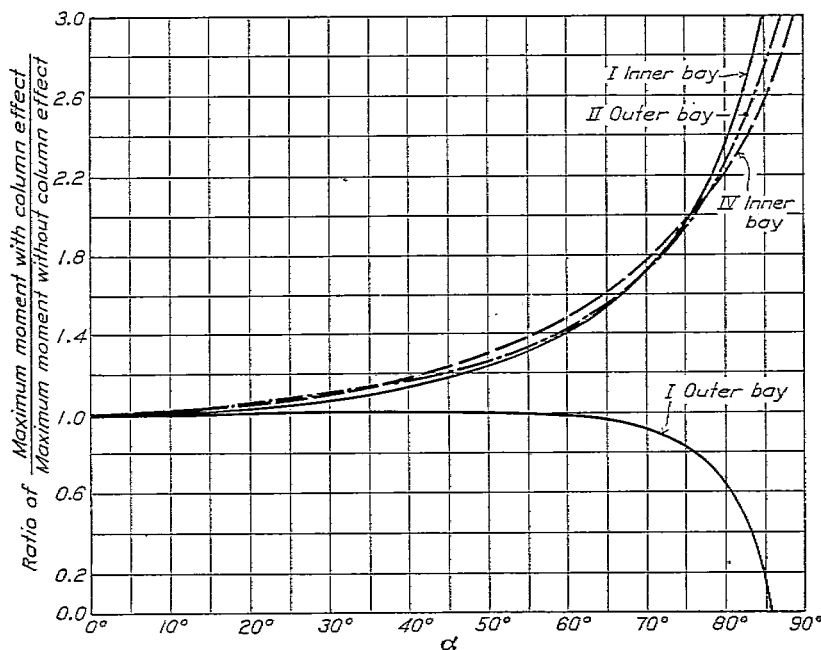

FIG. 14

the deflections were computed and plotted, the maximum moments within both bays were figured for a number of different values of that function, and a few extra moments were figured, too, for Cases II and IV. The results are tabulated below, and the ratios of the moments of the corresponding figures with no compression in the spar are plotted in Figure 14.

MOMENTS

α	Inner bay	Outer bay	Outer bay	Inner bay
	Case I	Case I [1]	Case II	Case IV
0°_____	75	66	191	104
15._____	76	66		
30._____	81	67	213	115
45._____	89	67		
60._____	106	65	273	154
75._____	146	55		
80._____	177	42	432	231

[1] The outer bay having the smaller value of α, these are the outer-bay moments for the condition in which α in the inner bay is as indicated in the left-hand column. In the outer bay it is actually about 10 per cent smaller.

Although there is some crossing back and forth among the curves in Figure 14, all are of the same general type (except, of course, the one which is drawn for a noncritical bay and in which the correction factors drop off below unity). The effect of the change in sign of the deflection in the outer bay in Case I is clearly shown. The column effect is at first positive and almost exactly large enough to balance the effect of the shift of the whole moment curve by virtue of
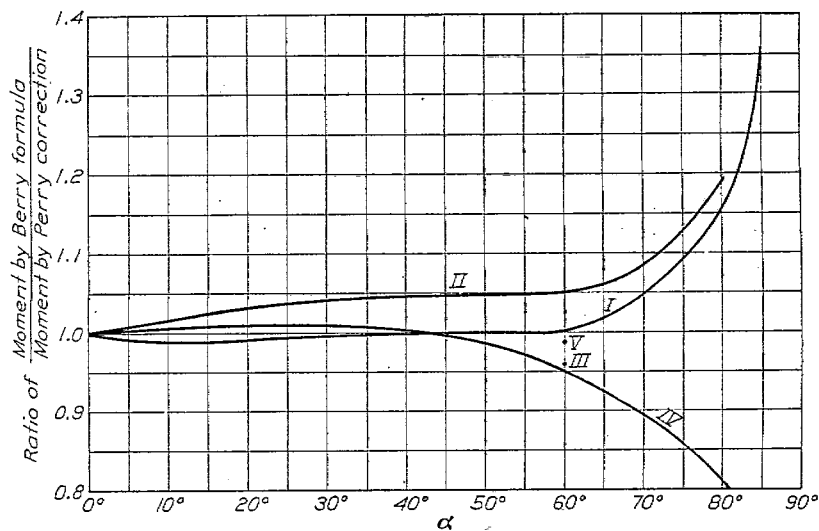

FIG. 15

the increasing absolute values of the moment at the inner strut, so that the maximum moment in the bay remains substantially constant over a wide range of values of α. Then, as α is further increased, the spar is forced down to a negative deflection as shown in Figure 12, the column effect becomes negative, and the moment in the bay drops off rapidly.

The application of the Perry formula may of course be extended in this case over the whole range of values of α, again taking n, the coefficient of shift of the points of inflection as 0.2. That has been done in Figure 15, whence it will be seen that in general, as might be expected, the errors arising from the use of an approximation grow larger and larger as the compressive load in the spar comes progressively to the "Euler load." The error is sometimes in one direction, sometimes in the other, and it becomes dangerously large when α exceeds 70° or 75°.

When α exceeds 65° in either bay, or when it has very nearly the same value in the two bays and that value is in excess of 45°, it is advisable that a direct calculation be made for the maximum moment in that bay in which α is largest, or in both if the two values are equal. The process of calculation is much facilitated by the use of charts giving M_B, as may best be shown by an example.

The inner bay in Case VII will be selected for purposes of illustration, and the work carried through with α equal to 60°, the figure already covered in the tabulation. The length of the inner bay in that case was 35 per cent of the total, that of the overhang 20 per cent, and the

bending moment at the middle support without column effect is found by the use of the three-moment equation or from the curves in Report No. 214 to be 152. This figure is, of course, based on a unit loading and a total length of 100 inches. Figure 11 indicates a correction factor of 1.13 to be applied to the moment when α is 60° and the length of the outer bay is 45 per cent, giving an approximate moment for M_B of 171. (It happens that the exact value of the moment with column effect has already been determined for this particular case, but the calculation will be carried through on the same basis as if that had not been done.)

A tabulation can now be made as for a representative single-bay spar, but in simplified form.

$$\alpha \text{_____} 60° (1.047 \text{ rad.})$$

$$\sec \alpha \text{_____} 2.000$$

$$\csc \alpha \text{_____} 1.155$$

$$M_c \text{_____} 0$$

$$M_B \text{_____} 171$$

$$\frac{1}{\mu^2}\left(=\frac{l^2}{4\alpha^2}\right) \text{_____} 279$$

$$\frac{M_B - M_c}{2} \text{_____} 85.5$$

$$\text{(A)} \left(\frac{M_B - M_c}{2}\right)\csc \alpha \text{_____} 98.6$$

$$\frac{1}{\mu^2} - \frac{M_B + M_c}{2} \text{_____} 193.5$$

$$\text{(B)} \left(\frac{1}{\mu^2} - \frac{M_B + M_c}{2}\right)\sec \alpha \text{_____} 387$$

$$\tan \mu x \left(=\frac{(A)}{(B)}\right) \text{_____} 0.255$$

$$\sec \mu x \ (=\sqrt{1 + \tan^2\mu x}) \text{_____} 1.0320$$

$$M_{max.} \left(= (B) \sec \mu x - \frac{1}{\mu^2}\right) \text{_____} 120$$

Working to a degree of accuracy of individual figures well within the scope of an ordinary slide-rule, the maximum moment can be calculated with an error of not more than 1 or 2 per cent in only three or four minutes more time than would be needed for the determination of the distance between the points of inflection and the application of Perry's formula or some other approximation of similar nature.

The only assumption involved in the work just done lay in the use of Figure 11 to find M_B. If the correction factor there plotted had been 5 per cent in error, however, so that M_B had been taken as 165 instead of 173, it would have changed the maximum calculated moment in the bay only from 120 to 126, introducing an error of about the same magnitude as that made in M_B itself.

BEST STRUT LOCATION

It was shown in Report No. 214, on the assumption that the Perry formula could be used to take care of the column correction in both bays, that the determining condition for maximum economy of material in an untapered two-bay spar is that the struts shall be so placed that the total resultant stresses at the inner strut, at the outer strut, and at the point of maximum moment in the inner bay are the same. That report also includes calculations of the best strut locations in terms of the relation between span, gap, and depth of spar, indicating that the best length of inner bay ranges from 32 to 40 per cent of the total length of spar, with overhangs correspondingly varying between 21 and 25 per cent. Figure 13 shows that an inner bay length of 32 per cent, combined with a 21 per cent overhang, makes α very slightly larger in the inner bay than in the outer, and any lengthening of the inner bay or shortening of the outer would, of course, increase the superiority of the inner bay α. The proportions recommended in the previous report would therefore make the inner bay the critical one for column effect in all cases in the light of the present analysis.

The Perry formula as used in the earlier work has here been shown to give approximately correct results, but no allowance was there made for the increase of M_B with column effect.

Aside from the absence of such allowance, the conclusions reached in Report No. 214 about strut distribution will still hold, as the inner bay proved the critical one by both treatments, the approximate and the exact. If the change of M_B be taken roughly into account, however, it appears advantageous to increase the length of overhand about 1 per cent while leaving the position of the inner strut unchanged. The curves of strut distribution in Report No. 214 might therefore well be modified by increasing the figure attached to each curve of equal overhand by one unit, making the best arrangement, for a truss of average aspe t ratio and gap-chord ratio, an inner bay of 34 per cent, an outer bay of 43, and an overhand of 23. These figures are given for a value of α of 60° in the inner bay, which is about the mean for two-bay biplanes. Any increase in that figure makes a still further increase of overhang advisable.

APPLICATION TO SPARS FIXED AT ENDS

It would be expected that the fixing moment at the inner end of a spar rigidly fixed to center section or fuselage would vary with column effect in the same general way as the moment in the inner bay, for if an increase of compression increases the upward deflection in the inner bay it must increase the curvature of the center line of the spar, and so the bending moments, both in the middle of that bay and at its inner end. It would also be anticipated that the column effect on the magnitudes of mid-bay moments would be less when the end of the spar is fixed than when it is pinned and free to change its slope. The single illustrative calculation made, bearing on a spar of the same proportions as that in Case VII but with a fixed end, has verified that expectation, for when α is 60° in the inner bay and bending moments vary in the manner shown by the tabulation below.

Moments	Inner end pinned			Inner end fixed		
	No column effect	$\alpha=60°$	Ratio	No column effect	$\alpha=60°$	Ratio
Inner strut	153	173	1.13	134	145	1.08
Inner end				86	88	1.02
Inner bay	85	119	1.40	43	28	.65

It is impossible to arrive at any very definite conclusion in the absence of more computations, but it seems probable that column effect on the fixing moment at the end of the spar will be of little importance in two-bay spars of usual dimensions, while the correction factor to be applied to that moment in a single-bay spar would always be less than the corresponding factor for the mid-bay moment in the same spar. It is probable, too, that the correction factor for the moment in the inner bay of a two-bay spar fixed at the end will be less than unity except when the value of α in the inner bay exceeds that in the outer by at least 20 per cent, and that in any case, whatever the ratio between those values may be, the use of the Perry formula without any shift of the points of inflection calculated under lateral load alone (that is, with the coefficient n taken equal to zero) will be a safe procedure.

REFERENCES

No. 1. NEWELL, J. S. "The Investigation of Structural Members under Combined Axial and Transverse Loads, Sec. I." Air Service Information Circular No. 493. Dec. 1, 1924.

No. 2. BERRY, ARTHUR. "Calculation of Stresses in Aeroplane Wing Spars." Transactions of the Royal Aeronautical Society No. I.

PIPPARD, A. J. S. and PRITCHARD, J. L. "Aeroplane Structures." Longmans, Green and Co. 1919. Pp. 116–120. (Abbreviated form of proof.) (Modified form dependent on different coordinates given in Reference 1.) (A similar method was developed in Germany during the war of 1914–1918.)

No. 3. WARNER, EDWARD P. "Wing Spar Stresses and Stress Charts." National Advisory Committee for Aeronautics Technical Report No. 214. 1925.

No. 4. MORLEY, A. "Strength of Materials."

No. 5. WARNER, EDWARD P. "Notes on Wing Spar Design." Aviation, May 20, 1922.